Gravitation and Electricity

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According to Riemann [1] geometry is based on the following two facts:

1. *Space is a three-dimensional continuum*, the manifold of its points is therefore represented in a smooth manner by the values of three coordinates x_1, x_2, x_3.

2. (*Pythagorean Theorem*) The square of the distance between two infinitesimally separated points

$$P = (x_1, x_2, x_3) \qquad \text{and} \qquad P' = (x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) \tag{1}$$

is (in any coordinate system) a quadratic form in the relative coordinates dx_i:

$$ds^2 = \sum_{ik} g_{ik} dx_i dx_k \qquad (g_{ik} = g_{ki}) \tag{2}$$

We express the second fact briefly by saying: the space is a *metrical* continuum. In the spirit of modern local physics we take the pythagorean theorem to be strictly valid only in the infinitesimal limit.

Special relativity leads to the insight that *time* should be included as a fourth coordinate x_o on the same footing as the three space-coordinates, and thus the stage for physical events, *the world*, is a *four-dimensional, metrical continuum*. The quadratic form (2) that defines the world-geometry is not positive-definite as in the case of three-dimensional geometry, but has positive-index 3. Riemann already expressed the idea that the metric should be regarded as something physically meaningful since it manifests itself as an effective force for material bodies, in centrifugal forces for example, and that one should therefore take into account that it interacts with matter; whereas previously all geometers and philosophers believed that the metric was an intrinsic property of the space, independent of the matter contained in it. It was on the basis of this idea, for which the possibility of fulfillment was not available to Riemann, that in our time Einstein (independently of Riemann) erected the grandiose structure of general relativity. According to Einstein the phenomena of gravitation can be attributed to the world-metric, and the laws through by which matter and metric interact are nothing but the laws of gravitation; the g_{ik} in (2) are the components of the gravitational potential.–Whereas the the gravitational potentials are the components of an invariant quadratic differential form, *electromagnetic phenomena* are controlled by a four-potential, whose components ϕ_i are the components of an invariant *linear* differential form $\sum \phi_i dx_i$. However, both phenomena, gravitation and electricity, have remained completely isolated from one another up to now.

From recent publications of Levi-Civita [2], Hessenberg [3] and the author [4] it has become evident that a natural formulation of Riemannian geometry is based on the concept of infinitesimal parallel-transfer. If P and P' are two points connected by a curve then one can can transfer a vector from P to P' along the curve keeping it parallel to itself. However, the transfer of the vector from P to P' is, in general, not integrable i.e. the vector that is obtained at P' depends on the path. Integrability holds only for Euclidean ('gravitation-free') geometry. — But in the Riemannian geometry described above there is contained a residual element of rigid geometry—with no good reason, as far as I can see; it is due only to the accidental development of Riemannian geometry from Euclidean geometry. The metric (2) allows the magnitudes of two vectors to be compared, not only at the same point, but at any two arbitrarily seperated points. *A true infinitesimal geometry should, however, recognize only a principle for transferring the magnitude of a vector to an infinitesimally close point* and then, on transfer to an arbitrarily distant point, the integrability of the magnitude of a vector is no more to be expected than the integrability of its direction. On the removal of this inconsistency there appears a geometry that, surprisingly, when applied to the world, *explains not only the gravitational phenomena but also the electrical.* According to the resultant theory both spring from the same source, indeed *in general one cannot seperate gravitation and electromagnetism in an arbitrary manner.* In this theory *all physical quantities have a world-geometrical meaning; the action appears from the beginning as a pure number. it leads to an essentially unique universal law; it even allows us to understand in a certain sense why the world is four-dimensional.* —I shall first sketch the construction of the corrected Riemannian geometry without any reference to physics; the physical application will then suggest itself.

In a given coordinate system the coordinates dx_i of a point P' relative to an infinitesimally close point P are the components of the *infinitesimal translation* $\overrightarrow{PP'}$ — see (1). The change from one coordinate-system to another is expressed by the continuous transformation:

$$x_i = x_i(x_1^* x_2^* \dots x_n^*) \qquad (i = 1, 2, \dots, n)$$

which determines the connection between the coordinates of the same point in the different systems. For the components dx_i and dx_i^* of the same infinitesimal translation of the point P we then have the linear transformation

$$dx_i = \sum_k \alpha_{ik} dx_k^* \tag{3}$$

in which the the α_{ik} are the values of the derivatives $\partial x_i/\partial x_k^*$ at the point P. A (contravariant) *vector* at the point P has n numbers ξ^i as components in every coordinate system, and on transforming the coordinates, these numbers transform in the same way as the infinitesimal translations in (3). I shall call the set of all vectors at P the *vector-space* at P. It is 1. *linear or affine* i.e. it is invariant with respect to the multiplication of a vector by a number and the addition of two vectors, and 2. *metrical*: by means of the symmetric bilinear form (2) an invariant scalar product

$$\chi.\eta = \eta.\chi = \sum g_{ik}\chi^i\eta^k$$

is defined for each pair of vectors χ, η. However, according to our point of view *this form is determined only up to an arbitrary positive proportionality-factor*. If the manifold is described by the coordinates x_i only the ratios of the components g_{ik} are determined by the metric at P. Physically also, only the ratios of the g_{ik} have a direct physical meaning. For a given point P the neighbouring points P' which can receive light-signals from P satisfy the equation

$$\sum_{ik} g_{ik}dx_i dx_k = 0$$

For the purpose of analytical representation we have to 1. choose a coordinate system and 2. in each point P determine the arbitrary proportionality-factor of the g_{ik}. Correspondingly each formula must have a double-invariance: 1. it must be *invariant with respect to arbitrary smooth coordinate transformations* 2. it must remain unchanged *when the g_{ik} are replaced by λg_{ik}* where λ is an arbitrary smooth function of position. Our theory is characterized by the appearance of this second invariance property.

An affine or linear map of the vector space at the point P onto the vector space at the point P^* is defined as the map $A \rightarrow A^*$ such that $\alpha\chi \rightarrow \alpha\chi^*$ and $\chi + \eta \rightarrow \chi^* + \eta^*$, where α is an arbitrary number. In particular the map is said to be is said to be a *similarity* map if the inner-product $\chi^*.\eta^*$ is proportional to the inner-product $\chi.\eta$ for all pairs of vectors χ and η. (Only this concept of similar maps has an objective meaning in our context; the previous theory allowed one to introduce the sharper concept of *congruent* maps.) The *parallel-transfer of a vector* at P to a neighbouring point P' is defined by the following two axioms:

1. The parallel transfer of the vectors at P to vectors at P' defines a similarity map.

2. If P_1 and P_2 are two neighbouring points to P and if the infinitesimal vectors $\overrightarrow{PP_2}$ and $\overrightarrow{PP_1}$ become $\overrightarrow{P_1P_{12}}$ and $\overrightarrow{P_2P_{21}}$, on parallel-transfer to P_2 and P_1 respectively then P_{12} and P_{21} coincide (commutativity).

The part of the first axiom that says that the parallel-transfer is an affine transformation of the vector space from P to P' is expressed analytically as follows: the vector ξ^i at $P = (x_1 x_2 \ldots x_n)$ is transferred to the vector

$$\xi^i + d\xi^i \quad \text{at} \quad P' = (x_1 + dx_1, x_2 + dx_2, \ldots, x_n + dx_n)$$

whose components are linear in ξ^i:

$$d\xi^i = -\sum_r d\gamma_r^i \xi^r \tag{4}$$

The second axiom requires that the $d\gamma_r^i$ are linear differential forms:

$$d\gamma_r^i = \sum_s \Gamma_{rs}^i dx_s,$$

whose coefficients have the symmetry property

$$\Gamma_{sr}^i = \Gamma_{rs}^i \tag{5}$$

If two vectors ξ^i, η^i at P are parallel-transferred to the vectors $\xi^i + d\xi^i$, $\eta^i + d\eta^i$ at P' the part of axiom 1 that goes beyond affinity to include similarity requires that

$$\sum_{ik} (g_{ik} + dg_{ik})(\xi^i + d\xi^i)(\eta^k + d\eta^k) \quad \text{and} \quad \sum_{ik} g_{ik} \xi^i \eta^k$$

are proportional. If we call the proportionality factor, which is infinitesimally close to unity, $(1 + d\phi)$ and define the lowering of indices in the usual manner as

$$a_i = \sum_k g_{ik} a^k$$

we then have

$$dg_{ik} - (d\gamma_{ki} + d\gamma_{ik}) = g_{ik} d\phi \tag{6}$$

From this it follows that $d\phi$ is a linear differential form:

$$d\phi = \sum_i \phi_i dx_i \tag{7}$$

If it is known, then the quantities Γ are determined by equation (6) or

$$\Gamma_{i,kr} + \Gamma_{k,ir} = \frac{\partial g_{ik}}{\partial x_r} - g_{ik} \phi_r$$

and the symmetry property (5). *The metrical connection of the space depends not only on the quadratic form (2)* (which is determined only up to a proportionality factor) *but on the linear form (7).* If, without changing coordinates, we replace g_{ik} by λg_{ik} the quantities $d\gamma^i_k$ remain unchanged, the $d\gamma_{ik}$ acquire a factor λ and dg_{ik} becomes $\lambda dg_{ik} + g_{ik}d\lambda$. Equation (6) then shows that $d\phi$ becomes

$$d\phi + \frac{d\lambda}{\lambda} = d\phi + d(\ln\lambda).$$

For the linear form $\phi_i dx_i$ the arbitrariness takes the form of an *additive total differential* rather than a proportionality factor that would be determined by a choice of scale. For the analytic representation of the geometry the forms

$$g_{ik}dx_i dx_k \qquad \phi_i dx_i \tag{8}$$

are on the same footing as

$$\lambda g_{ik}dx_i dx_k \quad \text{and} \quad \phi_i dx_i + d(\ln\lambda), \tag{9}$$

where λ is an arbitrary function of position. *The invariant quantity is therefore the anti-symmetric tensor with components*

$$F_{ik} = \frac{\partial\phi_i}{\partial x_k} - \frac{\partial\phi_k}{\partial x_i} \tag{10}$$

i.e. the form

$$F_{ik}dx_i\delta x_k = \frac{1}{2}F_{ik}\Delta x_{ik},$$

which depends bilinearly on two arbitrary translations dx and δx at the point P or, more precisely, on the surface-element

$$\Delta x_{ik} = dx_i\delta x_k - dx_k\delta x_i$$

determined by these two translations. The special case for which the magnitude of a vector at an arbitrary initial point can be parallel-transferred throughout the space in a path-independent manner appears when the g_{ik} can be chosen in such a way that the ϕ_i vanish. The Γ^i_{rs} are then nothing but the Christoffel 3-index symbols. The neccessary and sufficient condition for this to be the case is the vanishing of the tensor F_{ik}.

Accordingly, it is very suggestive to interpret ϕ_i as the electromagnetic potential and the tensor F as the *electromagnetic field*. Indeed, the absence of an electromagnetic field is the condition for the validity of Einstein's gravitational theory. If one accepts this interpretation one sees that electromagnetic quantities are such that their characterization by numbers in a given coordinate system is independent of the scale. In this theory one must adopt a new approach to the question of of scales and dimensions. Previously one spoke of a tensor being of second rank when, *after making an arbitrary choice of scale*, it was represented in every coordinate system by a matrix a_{ik} whose entries were the coefficients of an invariant bilinear form of two arbitrary independent infinitesimal translations

$$a_{ik} dx_i \delta x_k \tag{11}$$

Here we talk of a tensor when, having fixed a coordinate system and *making a definite choice of the proportionality factor of the* g_{ik}, the components a_{ik} are uniquely determined and indeed are determined in such a way that the form (11) is invariant with respect to coordinate transformations, but a_{ik} changes to $\lambda^e a_{ik}$ when g_{ik} changes to λg_{ik}. We say then that the tensor has *weight* e or, if a 'scale' l is assigned to the line-element ds, that it has dimension l^{2e}. The absolute invariant tensors are only those of weight zero. The field-tensor with the components F_{ik} is of this kind. According to (10) it satisfies the first system of Maxwell equations

$$\frac{\partial F_{kl}}{\partial x_i} + \frac{\partial F_{li}}{\partial x_k} + \frac{\partial F_{ik}}{\partial x_l} = 0$$

Once the concept of parallel-transfer is defined the geometry and tensor calculus is easily deduced.

a) *Geodesics.* Given a point P and a vector at P, the geodesic originating at P in the direction of this vector is obtained by continuously parallel-transferring the vector in its own direction. The differential equation for the geodesic takes the form

$$\frac{d^2 x_i}{d\tau^2} + \Gamma^i_{rs} \frac{dx_r}{d\tau} \frac{dx_s}{d\tau} = 0$$

for a suitable choice of the parameter τ. (It cannot, of course, be interpreted as the line of shortest length since the concept of length along a curve is not meaningful.)

b) *Tensor Calculus.* For example, to obtain a tensor-field of rank 2 from a covariant tensor-field of rank 1 and weight zero and components f_i by differentiation, we take any vector ξ^i at the point P with coordinates x_i, construct the invariant $f_i \xi^i$ and compute its infinitesimal variation on parallel-transfer to a neighbouring point P' with coordinates $x_i + dx_i$. We obtain

$$\frac{\partial f_i}{\partial x_k} \xi^i dx_k + f_r d\xi^r = \left(\frac{\partial f_i}{\partial x_k} - \Gamma^r_{ik} f_r \right) \xi^i dx_k.$$

The quantities in brackets on the right-hand side are the components of a tensor of rank 2 and weight zero which has been derived from the field f in a fully invariant manner.

c) *Curvature.* To construct the analogue of the Riemann tensor consider the infinitesimal parallelogram consisting of the points P, P_1, P_2 and $P_{12} = P_{21}$. Since the points P_{12} and P_{21} coincide, it makes sense to compute the difference between the vectors obtained at this point by taking any vector $\xi = \xi^i$ and parallel-transferring it to P_{12} via P_1 and P_2 respectively. For its components one obtains

$$\Delta\xi^i = R^i_j \xi^j, \tag{12}$$

where the R^i_j are independent of the vector ξ but depend linearly on the surface-element spanned by the two infinitesimal transfers $\overrightarrow{PP_1} = (dx_i)$ and $\overrightarrow{PP_2} = (\delta x_i)$:

$$R^i_j = R^i_{jkl} dx_k \delta x_l = \frac{1}{2} R^i_{jkl} \Delta x_{kl}.$$

The curvature components R^i_{jkl}, which depend only on the point P, have the following two symmetry properties: 1. they change sign on permutation of the last indices k and l; 2. if one cyclically permutes the indices j, k, l and adds, the sum is zero. If the index i is lowered we obtain in R_{ijkl} the components of a covariant tensor of 4th rank and weight 1. One sees by inspection that R splits in an invariant manner into two parts

$$R^i_{jkl} = P^i_{jkl} - \frac{1}{2}\delta^i_j F_{kl} \qquad \delta^i_j = 1 \quad (i = j) \quad \delta^i_j = 0 \quad (i \neq j), \tag{13}$$

where P_{ijkl} is anti-symmetric in the indices i and j as well as k and l. Whereas the equations $F_{ik} = 0$ characterize the absence of an electromagnetic field i.e. a space in which the transfer of magnitude is integrable, one sees from (13) that $P^i_{jkl} = 0$ are the invariant conditions for the absence of a gravitational field i.e. for the parallel transfer of directions to be integrable. Only in Euclidean space is there neither electromagnetism nor gravitation.

The simplest invariant of a linear map like (12) that assigns a vector $\delta\xi$ to every ξ is the trace

$$\frac{1}{n} R^i_i.$$

For this we obtain from (13) the form

$$-\frac{1}{2} F_{ik} dx_i \delta x_k,$$

which we have already encountered. The simplest invariant that can be constructed from a tensor of the form $-F_{ik}/2$ is the square of its magnitude

$$L = \frac{1}{4} F_{ik} F^{ik}.$$

Since the tensor F has weight zero, L is clearly an invariant of weight -2.

If g is the negative determinant of the g_{ik} and

$$d\omega = \sqrt{g}dx_o dx_1 dx_2 dx_3 = \sqrt{g}dx$$

is the infinitesimal volume element, then, as is well-known, the Maxwell theory is determined by the electromagnetic Action, which is equal to the integral $\int Ld\omega$ over an arbitrary volume of this simplest invariant, in such a way that for arbitrary variations of the g_{ik} and ϕ_i which vanish on the boundary we have

$$\delta \int Ld\omega = \int \left(s^i \delta\phi_i + T^{ik}\delta g_{ik}\right) d\omega$$

where

$$s^i = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}F^{ik}\right)}{\partial x_k}$$

are the left-hand side of the Maxwell equations (on the right-hand side of which is the electromagnetic current) and the T^{ik} are the components of the energy-momentum tensor of the electromagnetic field. Since L is an invariant of weight -2 and the volume element an invariant of weight $\frac{n}{2}$ the integral $\int Ld\omega$ then has a meaning only when the dimension is $n = 4$. *Thus in our context the Maxwell equations are possible only in 4 dimensions.* But in four dimensions the electromagnetic action is a pure number. Its magnitude in CGS units can, of course, only be determined when a computation based on our theory is applied to a physical problem such as the electron.

Passing on from Geometry to Physics, we have to assume, following the example of Mie's theory [5], that the whole set of natural laws is based on a definite integral-invariant, the action

$$\int Wd\omega = \int \mathcal{W}dx \qquad (\mathcal{W} = W\sqrt{g})$$

in such a way that *the actual world is selected from the class of all possible worlds by the fact that the Action is extremal in every region* with respect to the variations of the g_{ik} and ϕ_k which vanish on the boundary of that region. W, the action-density, must be an invariant of weight -2. *The action is in any case a pure number*; in this way our theory gives pride of place to that part of atomic theory that is the most fundamental according to modern ideas: the action. The simplest and most natural Ansatz that we can make for W is

$$W = R^i_{jkl} R_i^{jkl} = |R|^2 \tag{14}$$

According to (13) this can be written as

$$W = |P|^2 + 4L$$