Radiance Theory

Projective ray geometry. Radiance in geometric optics
Projective space treatment of lightfields

1. The world will be 2D instead of 3D.
2. We will need to use points at actual infinity.
   Projective geometry.

**Projective points**

Consider a 3D vector space. Any vector \( \mathbf{v} = (x, y, z) \) defines a 1D subspace:

That is a class of equivalence: \( [x, y, z] = \{ (\lambda x, \lambda y, \lambda z) \text{ for all } \lambda \neq 0 \} \)

\( q = [x, y, z] \) is called *projective point*. 
For $z \neq 0$

$q = [x, y, z] = [x/z, y/z, 1]$

For $z = 0$

$p = [x, y, 0]$ is at infinity

We can handle actual infinity!
Projective space treatment of lightfields

Projective lines

Consider a 3D vector space. Any 2 vectors \( \mathbf{a} = (x, y, z) \) and \( \mathbf{b} = (u, v, w) \) define a plane: They form a basis of a 2D subspace. Described by a bivector \( \mathbf{a} \wedge \mathbf{b} \). It is a “double vector” linear by both \( \mathbf{a} \) and \( \mathbf{b} \). This anti symmetric tensor represents the line \( \mathbf{L} \). In matrix form

\[
\mathbf{L} = \begin{bmatrix}
0 & xv - yu & xw - zu \\
yu - xv & 0 & yw - zw \\
zu - xw & zw - yu & 0
\end{bmatrix}
\]
Projective lines

Consider a 3D vector space. Any 2 vectors \( \mathbf{a} = (x, y, z) \) and \( \mathbf{b} = (u, v, w) \) define a plane described by a bivector \( \mathbf{a} \wedge \mathbf{b} \). Any other 2 vectors in the same plane produce a bivector proportional to \( \mathbf{a} \wedge \mathbf{b} \). It defines the plane.

The cross (vector) product \( \mathbf{a} \times \mathbf{b} = *(\mathbf{a} \wedge \mathbf{b}) = (yw - zv, zu - xw, xv - yu) \), where \( * \) is the Hodge operator. Gives orthogonal completion to a subspace.
Projective space treatment of lightfields

Summary:

Vectors and bivectors (geometrically lines and planes) represent points and lines in projective space. This includes points at infinity and the line at infinity.

A plane is also represented by the vector orthogonal to it, the cross product.

Two points define a line: The bivector $a \wedge b$ defines a plane (projective line).

Two lines intersect at a point: The two cross products are two vectors. Their cross product is a vector defining the intersection point projectively.
Projective space treatment of lightfields

Two points define a line

Two lines define a point.
Projective space treatment of lightfields

Interpretation

Light rays (as projective lines) can be defined by 2 points: The line is the cross product or the bivector of those 2 points.

Given a point from a ray, the ray is completely defined by direction, i.e. point at infinity. A finite point and a point at infinity define a ray uniquely.

No rays are excluded. No limitation to angle / field of view.

Next we will be using the more popular 2-plane parametrisation that is easier, but cannot handle certain rays.

A true model needs to be 3D projective space.
Ray Transforms

The main laws of geometric optics
Two Parameterizations of Rays

Two-Plane

Point-Angle

$q \rightarrow q'$

$q \rightarrow p = \text{slope}$

optical axis

optical axis
Transport Through Space

- Ray travels distance $t$ through space

- $q$ and $p$ are transformed to $q'$ and $p'$:

  $$ q' = q + tp $$
  $$ p' = p $$

- In matrix notation:

  $$ \begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = T \begin{bmatrix} q \\ p \end{bmatrix} $$
Lens Transformation

- Ray is refracted at a thin lens
- “The further from center, the more refraction”:

\[
q' = q \\
p' = p - \frac{1}{f}q
\]

\[
\begin{bmatrix}
q' \\
p'
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{bmatrix} \begin{bmatrix}
q \\
p
\end{bmatrix} = L \begin{bmatrix}
q \\
p
\end{bmatrix}
\]
Summary: Two Primary Optical Transforms

Transport

\[
\begin{bmatrix}
q' \\
p'
\end{bmatrix} = T \begin{bmatrix}
q \\
p
\end{bmatrix}
\]

\[
T = \begin{bmatrix}
1 & t \\
0 & 1
\end{bmatrix}
\]

Lens

\[
\begin{bmatrix}
q' \\
p'
\end{bmatrix} = L \begin{bmatrix}
q \\
p
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{bmatrix}
\]
Phase Space

- This is simply the \((q, p)\) space of rays. It is a 4D vector space with zero vector the optical axis.

- Each ray is a 4D point (a vector) in that space.

- Any optical device, like a microscope or a telescope, is a matrix that transforms an incoming ray into an outgoing ray.

- This matrix can be computed as a product of the optical elements that make up the device.
Transformations in Phase Space

Space transport

Lens refraction
Composition of Optical Elements

- Transformations corresponding to compositions of optical elements are determined by the constituent transformations.

Consider a system with transport $T_1$, lens $L_f$ and transport $T_2$.

- What is $\begin{bmatrix} q''' \\ p'''' \end{bmatrix}$ in terms of $\begin{bmatrix} q \\ p \end{bmatrix}$?
Composition of Optical Elements

- Consider one element at a time
- What is $\begin{bmatrix} q' \\ p' \end{bmatrix}$ in terms of $\begin{bmatrix} q \\ p \end{bmatrix}$?

Transport by $T_1$

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = T_1 \begin{bmatrix} q \\ p \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 1 & t_1 \\ 0 & 1 \end{bmatrix}$$
Composition of Optical Elements

- Consider one element at a time
- What is \[ \begin{bmatrix} q'' \\ p'' \end{bmatrix} \] in terms of \[ \begin{bmatrix} q \\ p \end{bmatrix} \]?

\[ \begin{bmatrix} q' \\ p' \end{bmatrix} \begin{bmatrix} q'' \\ p'' \end{bmatrix} \]

- Lens transform by \( L_f \)
\[ \begin{bmatrix} q'' \\ p'' \end{bmatrix} = L_f \begin{bmatrix} q' \\ p' \end{bmatrix} \]

- Substitute for \[ \begin{bmatrix} q' \\ p' \end{bmatrix} \]
\[ \begin{bmatrix} q'' \\ p'' \end{bmatrix} = L_f T_1 \begin{bmatrix} q \\ p \end{bmatrix} \]
Composition of Optical Elements

- Consider one element at a time
- What is $\begin{bmatrix} q'''' \\ p'''' \end{bmatrix}$ in terms of $\begin{bmatrix} q \\ p \end{bmatrix}$?

- Transport by

$$\begin{bmatrix} q'''' \\ p'''' \end{bmatrix} = T_2 \begin{bmatrix} q'' \\ p'' \end{bmatrix}$$

- Substitute for $\begin{bmatrix} q'' \\ p'' \end{bmatrix}$

$$\begin{bmatrix} q'''' \\ p'''' \end{bmatrix} = T_2 L_f T_1 \begin{bmatrix} q \\ p \end{bmatrix}$$
In-Class Exercise

- Three-lens system
- Composition: \[ A = T_4 L_3 T_3 L_2 T_2 L_1 T_1 \]
Principal Planes

- Gauss discovered that the matrix for any optical transform can be written as a product of some appropriate translation, lens, and translation again.
- Often expressed as “principal planes” (green):
Principal Planes

- No constraint is placed on the position of the principal planes of the focal length; no travel between principal planes.
Traditional Camera

Transfer matrix:

\[
A = \begin{bmatrix}
1 & b \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{bmatrix}
\begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix}
\]
Traditional Camera

\[ A = \begin{bmatrix}
1 & b \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{bmatrix} \begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix} \]

\[ = \begin{bmatrix}
1 & b \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & a \\
-\frac{1}{f} & 1 - \frac{a}{f}
\end{bmatrix} = \begin{bmatrix}
1 - \frac{b}{f} & ab \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{f}\right) \\
-\frac{1}{f} & 1 - \frac{a}{f}
\end{bmatrix} \]
Traditional Camera

How do we focus?

\[ A = \begin{bmatrix} 1 - \frac{b}{f} & ab \left( \frac{1}{a} + \frac{1}{b} - \frac{1}{f} \right) \\ -\frac{1}{f} & 1 - \frac{a}{f} \end{bmatrix} \]
Traditional Camera

How do we focus?

Make top-right element to be zero

\[
A = \begin{bmatrix}
1 - \frac{b}{f} & ab \left( \frac{1}{a} + \frac{1}{b} - \frac{1}{f} \right) \\
-\frac{1}{f} & 1 - \frac{a}{f}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 - \frac{b}{f} & 0 \\
-\frac{1}{f} & 1 - \frac{a}{f}
\end{bmatrix}
\]
We enforce this condition by:

\[
\frac{1}{a} + \frac{1}{b} - \frac{1}{f} = 0
\]
Traditional Camera

We have derived the lens equation:

\[
\frac{1}{a} + \frac{1}{b} = \frac{1}{f}
\]

\[
A = \begin{bmatrix}
-\frac{b}{a} & 0 \\
-\frac{1}{f} & -\frac{a}{b}
\end{bmatrix}
\]
In-Class Exercise

What is $\det(A)$?

Answer: $\det(A) = 1$

$$A = \begin{bmatrix} 1 - \frac{b}{f} & ab \left( \frac{1}{a} + \frac{1}{b} - \frac{1}{f} \right) \\ -\frac{1}{f} & 1 - \frac{a}{f} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$
In-Class Exercise

What is $\det(A)$?

$\det(A) = 1$

$$A = \begin{bmatrix} -\frac{b}{a} & 0 \\ -\frac{1}{f} & -\frac{a}{b} \end{bmatrix}$$
“2F” Camera

- Three optical elements: space, lens, space

- Transformation: $A = T_f L_f T_f$
In-Class Exercise

\[ A = \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{f} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & f \\ \frac{-1}{f} & 0 \end{bmatrix} \]

What is \( \text{det}(A) \)?

Again we compute \( \text{det}(A) = 1 \)
In-Class Exercise

- In two different cases (conventional and "2F" camera) we get the same result: $\det(A) = 1$
- Is that always the case?
- Hint: Every optical system is a composition of $L$ and $T$, which both have $\det = 1$
- And the determinant of a product is the product of the determinants.
- This is an important physical property.
Radiance

Definition and main mathematical properties
Conservation of Volume

- For the 2 transforms, the 4D box changes shape.
- Volume remains the same (shear).
- Must remain the same for any optical transform!

\[ L = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \]

\[ T = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]
Conservation of Radiance

- Radiance is energy density in 4D ray-space
- Energy is conserved; volume is conserved
- Radiance = (energy) / (volume)
- Radiance is also conserved!
- “Radiance is constant along each ray”
Additional Notes on Conservation of Radiance

Similar principle in Hamiltonian mechanics in terms of coordinate $q$ and momentum $p$: Liouville’s theorem

As the system evolves in time, volume in qp-space is conserved

- State space and particle systems
- Quantum mechanics

In optics, astronomy, and photography, radiance conservation is often mentioned (or implied) in relation to:

- Throughput
- Barlow lens
- Teleconverter
- F/number
Additional Notes on Conservation of Radiance

- Optical state space is a vector space with the optical axis being the zero vector
  - Optical devices, like cameras and microscopes perform linear transforms.
- Optical transforms are symplectic:
  - They preserve a skew-symmetric dot product in $qp$-space
  - In terms of that dot product each ray is orthogonal to itself
- For any optical transform $A$, $\det A = 1$
Radiance Transforms

- Optical elements transform rays
- They also transform radiance

- Points in ray space \( x = \begin{bmatrix} q \\ p \end{bmatrix} \)
- Radiance before optical transform \( r(x) \)
- Radiance after optical transform \( r'(x) \)
Radiance Transforms

\[ x' = Ax \]

Due to radiance conservation,

\[ r'(x') = r(x) \]

\[ r'(x') = r(A^{-1}x') \]

Since \( x' \) is arbitrary, we can replace it by \( x \)

\[ r'(x) = r(A^{-1}x) \]
Radiance Transforms

- The radiance after optical transformation is related to the original radiance by: \( r'(x) = r(A^{-1}x) \)

- What is that for translation?

\[
T = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}
\]

- So \( r'(q, p) = r(q - tp, p) \)
In-Class Exercise

- The radiance after optical transformation is related to the original radiance by: \( r'(x) = r(A^{-1}x) \)

- What is that for a lens?

\[
L = \begin{bmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{bmatrix} \quad L^{-1} = \begin{bmatrix}
\frac{1}{f} & 0 \\
1 & 1
\end{bmatrix}
\]

- So \( r'(q, p) = r(q, p + \frac{q}{f}) \)
Image Rendering

- Now that we have the lightfield (all of the light in a scene) – how do we turn \( q \) and \( p \) into a picture? (A rendered image)?
- Use physics of integral image formation
Image Rendering

- A traditional image is formed by integrating rays from all directions at each pixel.
- A traditional image is rendered from a radiance according to:

\[ I(q) = \int_{p} r(q, p) dp \]
Radiance Capture (Cameras)
Capturing Radiance

- To capture radiance, we need to capture rays from different directions individually.
- But sensors are not directional.
- Rays from different directions need to be mapped to different positions (different pixels).
Pinhole Camera

- Rays can only enter camera at one point \((q = 0)\)
- Rays from different directions spread apart inside camera
- And are captured at different positions on the sensor

Switches direction and position
- Captures angular distribution of radiance
Pinhole Camera

More precisely

\[ r'(q, p) = r(q, p)\delta(q) \]
\[ r_t(q, p) = r(q - tp)\delta(q - tp) \]

\[ I(q) = \int r_t(q, p) dp = \frac{1}{t} \int r(q - tp, \frac{tp}{t})\delta(q - tp)d(tp) = \frac{1}{t}r(0, \frac{q}{t}) \]

Switches angle and position

Captures angular distribution of radiance
“2F” Camera

- Generalizes pinhole camera
- Lens of focal length $f$ is placed at distance $f$ from sensor
- Switches angle and position
- Captures angular distribution of radiance assuming it doesn’t change much with $q$ (close to $q = 0$)
“2F” Camera

- This is the lens generalization of the pinhole camera
- Three optical elements: space, lens, space

Transformation: \[ A = T_f L_f T_f \]
“2F” Camera

- This is the lens generalization of the pinhole camera
- Three optical elements: space, lens, space

\[ A = \begin{bmatrix} 0 & f \\ -\frac{1}{f} & 0 \end{bmatrix} \]

Show that \( I(q) = \frac{D}{f} r(0, \frac{q}{f}) \)

- Switches angle and position
- Captures angular distribution of radiance (at \( q = 0 \))
Traditional 2D Camera

Three optical elements: space, lens, space

\[
A = \begin{bmatrix}
-\frac{b}{a} & 0 \\
-\frac{1}{f} & -\frac{a}{b}
\end{bmatrix}
\]

Show that \( I(q) = \frac{D}{b} r(-\frac{a}{b} q, 0) \) approximately.
Capturing Radiance

- Pinhole camera or “2F” camera capture an image $I(q)$
- $I(q)$ captures angular distribution of radiance
- Only for small area around $q = 0$ so far
- For complete radiance, we need to capture angular distribution for all $q$

Basic Idea: *Replicate pinhole or “2F” at every $q*

- Ives (pinhole)
- Lippmann (“2F”)
Ives’ Camera (based on the pinhole camera)

At the image plane:

Multiplexing in space:

Each pinhole image captures angular distribution of radiance. All images together describe the complete 4D radiance.
Lippmann’s Camera (based on 2F)

- Space multiplexing
- Lenses instead of pinholes
- A “2F camera” replaces each pinhole camera in Ives’ design
Camera Arrays

- The most popular lightfield camera is simply an array of conventional cameras, like the Stanford array.

- Alternatively, an array of lenses/prisms with a common sensor, like the Adobe array.
Adobe Array of Lenses and Prisms
Arrays of Lenses and Prisms

Shifting cameras from the optical axis means: We need to extend the vector space treatment to affine space treatment.

Prism transform

\[
\begin{pmatrix}
q' \\
p'
\end{pmatrix} = \begin{pmatrix}
q \\
p
\end{pmatrix} + \begin{pmatrix}
0 \\
\alpha
\end{pmatrix}
\]

Shifted lens

\[
\begin{pmatrix}
q' \\
p'
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{pmatrix} \begin{pmatrix}
q - s \\
p
\end{pmatrix} + \begin{pmatrix}
s \\
0
\end{pmatrix}
\]

Lens + prism

\[
\begin{pmatrix}
q' \\
p'
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{pmatrix} \begin{pmatrix}
q \\
p
\end{pmatrix} + \begin{pmatrix}
0 \\
\frac{s}{f}
\end{pmatrix}
\]
Radiance in the Frequency Domain

In the frequency domain, the two optical elements switch places: lens becomes space; space becomes lens.
Radiance Transforms (Frequency Domain)

- Converting radiance into frequency representation gives us a new tool for analysis, and new power.
- A pixel no longer stands by itself, representing a point in one single image / slice in 4D radiance.
- In the frequency domain one pixel can represent multiple images at the same time.
- Those images are slices of the 4D radiance, but now in the frequency domain.
- By optically combining multiple frequencies, we achieve new and more efficient use of the sensor.
Radiance Transforms (Frequency Domain)

Radiance in frequency representation:

\[ R(\omega) = \int r(x) e^{i\omega \cdot x} \, dx \]

where \[ \omega = \begin{bmatrix} \omega_q \\ \omega_p \end{bmatrix} \] and \[ \omega \cdot x = \omega_q q + \omega_p p \]

Next we derive the relation between \[ R'(\omega) \]
and \[ R(\omega) \] due to optical transform \[ x = Ax_0 \]
Radiance Transforms (Frequency Domain)

\[ R'(\omega) = \int r'(x)e^{i\omega \cdot x} \, dx \]

\[ = \int r(A^{-1}x)e^{i\omega \cdot x} \, dx \]

\[ = \int r(A^{-1}x)e^{i\omega \cdot AA^{-1}x} \, dx \]

\[ = \int r(x_0)e^{i\omega \cdot Ax_0} \, dx_0 \]

\[ = \int r(x_0)e^{iA^t\omega \cdot x_0} \, dx_0 \]

\[ = R(A^T\omega) \]
Radiance Transforms

Main results (summary):

\[ x = Ax_0 \]
\[ r'(x) = r(A^{-1}x) \]
\[ R'(\omega) = r(A^T\omega) \]

Note: Shear is in the other direction in frequency domain due to the transposed matrix. Lens <-> space.

Note: The inverse always exists because \( \det A = 1 \).
Ives’ Camera: Frequency Multiplexing

- Poisson summation formula

\[ \sum_{m=-\infty}^{\infty} \delta(q - m) = \sum_{n=-\infty}^{\infty} e^{in2\pi q} \]

“train of delta functions = train of frequencies”

Prove

\[ \sum_{m=-\infty}^{\infty} \delta(q - mb) = \frac{1}{b} \sum_{n=-\infty}^{\infty} e^{in2\pi \frac{q}{b}} \]
Ives’ Camera: Frequency Multiplexing

\[ r'(q, p) = r(q, p) \sum_{m=-\infty}^{\infty} \delta(q - mb) \]

\[ R'(\omega) = \int r(q, p) \sum_{m} \delta(q - mb) e^{i\omega \cdot x} \, dx \]

\[ = \frac{1}{b} \int r(q, p) \sum_{n} e^{in\frac{2\pi q}{b}} e^{i(\omega_q q + \omega_p p)} \, dq \, dp \]

Transposed translation \( f \)

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\omega_q \\
\omega_p
\end{pmatrix}
= 
\begin{pmatrix}
\omega_q \\
f\omega_q + \omega_p
\end{pmatrix}
\]

\[ R_f(\omega) = \sum_{n=-\infty}^{\infty} R(\omega_q + n\frac{2\pi}{b}, f\omega_q + \omega_p) \]
Ives’ Camera: Frequency Multiplexing

Band limited radiance

\[ R_f(\omega) = \sum_{n=-\infty}^{\infty} R(\omega_q + n \frac{2\pi}{b}, f\omega_q + \omega_p) \]

Veeraraghavan’s idea:
Cosine Mask Camera (MERL)

- A transparency, superposition of cos terms, is placed at distance $f$ from the sensor
- Consider for example:

$$\frac{1}{2} \left( 1 + \cos(\omega_0 q) \right)$$

Derive the expression for the radiance at the sensor:

$$R_f(\omega_q, \omega_p) = \frac{1}{2} R(\omega_q, f\omega_q + \omega_p)$$

$$+ \frac{1}{4} \left( R(\omega_q + \omega_0, f\omega_q + \omega_p) + R(\omega_q - \omega_0, f\omega_q + \omega_p) \right)$$
Periodic Mask Camera (Adobe)

Input:
F/5.6
Periodic Mask Camera (Adobe)

Output:
F/5.6
Periodic Mask Camera (Adobe)

Output:
F/5.6
Ives' camera: Multiplexing in frequency
Ives' camera: Multiplexing in frequency
Periodic Mask Camera (Adobe)

Output:
F/4
Periodic Mask Camera (Adobe)

Output:
F/4
“Mosquito Net” Camera

\[
\frac{1}{a} + \frac{1}{b} = \frac{1}{f}
\]

\[
\frac{da}{a^2} = -\frac{db}{b^2}
\]

\[
b = 80\text{mm} \quad a = 2\text{m} \quad da = 10\text{cm}
\]
“Mosquito Net Camera” Refocusing
Lippmann’s Camera

- Space multiplexing
- Lenses instead of pinholes
- A “2F camera” replaces each pinhole camera in Ives’ design
Lippmann’s Camera – “Heterodyning”

- Frequency multiplexing or “heterodyning” analysis can be done in two steps:

1. Consider array of shifted pinhole-prisms with constant shift $a$, and prism angle $a/f$

   ![Diagram of shifted pinhole-prisms](image)

2. Superposition of arrays with different shifts to implement microlenses as Fresnel lenses.
Lippmann’s Camera – “Heterodyning”

Starting with

\[ R(\omega) = \int r(q, p + \frac{a}{f}) \sum_{m} \delta(q - mb - a) e^{i\omega \cdot x} \, dx \]

Derive the radiance at the focal plane

Show that at zero angular frequency it becomes:

\[ R_f(\omega_q, 0) = \frac{1}{b} \sum_{n} e^{-i(\omega_q a + n \frac{2\pi a}{b})} R(\omega_q + n \frac{2\pi}{b}, f \omega_q) \]
Lippmann’s Camera – “Heterodyning”

\[
R'(\omega) = \int r(q, p) \sum_m \delta(q - mb) e^{i\omega \cdot x} dx
\]

\[
= \frac{1}{b} \int r(q, p) \sum_n e^{in \frac{2\pi q}{b}} e^{i(\omega_q q + \omega_p p)} dq dp
\]

\[
= \frac{1}{b} \sum_n R(\omega_q + n \frac{2\pi}{b}, \omega_p).
\]

\[
R_f(\omega) = \sum_{n=-\infty}^{\infty} R(\omega_q + n \frac{2\pi}{b}, f\omega_q + \omega_p)
\]
Lippmann’s Camera – “Heterodyning”

\[ R'(\omega) = \int r(q, p + \frac{a}{f}) \sum_m \delta(q - mb - a)e^{i\omega \cdot x} dx \]

\[ = \frac{1}{b} \int r(q, p + \frac{a}{f}) \sum_n e^{in\frac{2\pi(q-a)}{b}} e^{i(\omega q + \omega_p p)} dq dp \]

\[ = \frac{1}{b} \sum_n e^{-i(\omega_p \frac{a}{f} + n\frac{2\pi a}{b})} R(\omega_q + n\frac{2\pi}{b}, \omega_p). \]

\[ R_f(\omega) = \frac{1}{b} \sum_n e^{-i((f\omega_q + \omega_p) \frac{a}{f} + n\frac{2\pi a}{b})} R(\omega_q + n\frac{2\pi}{b}, f\omega_q + \omega_p) \]

\[ R_f(\omega_q, 0) = \frac{1}{b} \sum_n e^{-i(\omega_q a + n\frac{2\pi a}{b})} R(\omega_q + n\frac{2\pi}{b}, f\omega_q) \]
Lippmann’s Camera – “Heterodyning”

Plenoptic (Integral) camera with frequency multiplexing

Thanks to Ren Ng for providing the lightfield image.